Different Research Directions and Approaches to Interval Decision Analysis within the DECIDE Research Group

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Scope of Document

This document is a requirement for graduation at KTH and describes some of the major differences between, on one hand, my approach and, on the other hand, Per-Erik Malmnäs' and Love Ekenberg's approach to interval decision analysis. We all work in the same research group, DECIDE, and two of us have had the third as thesis supervisor, Per-Erik Malmnäs, so there are similarities. The research topic is similar – decision analysis using imprecise information – and the two different research directions were both inspired by Malmnäs' old interest in philosophical decision analysis.

Research Differences

My starting point was, to some extent, Malmnäs' work during the 1980s and early 1990s. This was also the starting point for Ekenberg and I was familiar with Malmnäs' and Ekenberg's work since I was, as a new Ph.D. student, the implementer of the **μ** decision solver software based on Malmnäs' and Ekenberg's results (1993). It was during that implementation that I realised that such a decision-analytic method must have rather different and more powerful features and I decided to make that my thesis topic [D97]. Some of the more important differences include:

- Constraints are not enough, especially since the decision maker is told to be deliberately imprecise – they will be too wide and the evaluation will have too much overlap in the sense that too many plausible solutions have different alternatives as the preferred one. The interpretation of constraints as 'negative' statements, cutting off excluded parts of the space, and the introduction of estimates as 'positive' statements solve this problem.
- The evaluation rules were not powerful enough, showing the 't' values for the alternatives separately. Also, the concept of proportion was flawed, as is discussed separately below. The introduction of the concepts of contraction and expansion rectifies this and makes the Delta method coincide with the ordinary expected value at full contraction, which is not the case with proportions.
- The evaluation algorithm in the **μ** solver was too slow and it covered only the case where a strict order was forced on all the values in each

alternative. Also, it required more preconditions than was realised at that time. The corrected algorithm is called VB-Opt in the thesis.[1] The main algorithm in the thesis is instead PB-Opt, covering a more realistic subset of possible constraint sets and admitting fast solving of a large class of decision problems. Also, the algorithms in **μ** were not formulated in terms of well-known procedures, such as for example LP formulations for consistency and hull calculations. Further, the bilinear elimination algorithm suggested by Malmnäs suffered from severe complexity and was not practically implementable. In my thesis, I suggest a solver hierarchy with good performance for the exact global solution and nice approximate behaviour for local anytime results (suitable for interactive applications).

More information on the differences is provided at the beginning of the thesis. In my thesis preface [D97] there is a section called Contributions, which tries to list my research findings without referring to flaws in Malmnäs' and Ekenberg's work, who are good friends and esteemed colleagues, since I have no desire to point out other's mistakes. However, because it is a requirement for graduation, and as a case in point, below is a description of one such problematic issue – the inadequacy of their concept of proportion and reduction procedure.

The Concept of Proportion

This section argues that Malmnäs' and Ekenberg's concepts of reduction and proportion are ineffectual in a method for decision analysis of interval and comparative statements.

Usually, different alternatives are superior in different parts of the consistent space. A procedure for comparing the volumes in which the respective alternatives dominate seems a reasonable way of finding out the "best" alternative. If an alternative is superior to another in a considerably larger part of the consistent space, then it is reasonable to call that alternative better. The ability to make volume estimates in the combined base $P \cup U$ is therefore important.

[1] "*Thesis*" in this document refers to my Ph.D. thesis manuscript, to be presented at a dissertation seminar on Monday, May 26, 1997, in Electrum, Kista.

Were it not for the high dimensionality of the consistent space for real-life decision problems, a Riemann integral could be calculated to measure the volumes of the consistent parts. Considering interactive use, however, the evaluation must resort to approximations. One suggestion is to make a (possibly weighted) Monte Carlo simulation [M90] but because of the high dimensionality, that would be an inefficient approach. Another suggestion is to measure the dominated regions indirectly by using the concept of proportion as an alternative volume measurement [E94]. Proportions apply to any consistent base, be it a probability, utility, or combined base.

The Reduction Procedure

Interval statements are imprecise and uncertain by nature. Hence it is natural to consider values near the boundaries of the intervals to be less reliable than more central values. By using the concepts of reduction and proportion this can be presented as follows.[2] First, there is a need for a procedure for studying decreasing volumes in the base.

> **Definition 1:** X is a base in the variables $x_1,\ldots,x_n$ and $d \in [0,1]$ is a real number. Further, $[a_i,b_i]$ are the intervals corresponding to the variables x_i in the orthogonal hull.
> Then a G(d)-reduction of X is to add the interval statements $\{i = 1,\ldots,n: x_i \in [a_i+d\cdot(b_i-a_i)/2,\ b_i-d\cdot(b_i-a_i)/2]\}$ to the base.[3]

Example 1: Suppose there is a probability base P with the following statements for alternative A_1:

$p_{11} \in [0.20, 0.60]$

$p_{12} \in [0.10, 0.30]$

$p_{13} \in [0.30, 0.50]$

To reduce the base by 50%, a G(0.5)-reduction of P yields

$p_{11} \in [0.20 + 0.5 \cdot(0.60-0.20)/2,\ 0.60 - 0.5 \cdot(0.60-0.20)/2] =[0.30, 0.50]$

$p_{12} \in [0.10 + 0.5 \cdot(0.30-0.10)/2,\ 0.30 - 0.5 \cdot(0.30-0.10)/2] = [0.15, 0.25]$

$p_{13} \in [0.30 + 0.5 \cdot(0.50-0.30)/2,\ 0.50 - 0.5 \cdot(0.50-0.30)/2] = [0.35, 0.45]$

From inspecting the resulting intervals, it is clear what a reduction does.

A more elaborate reduction algorithm that introduces the statements into X one at a time can be found in [E94] but it is shown there to be order-

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[2] Although the definitions in [E94] are slightly different and adapted here to fit this presentation, the resulting procedures are the same.

[3] A G(0)-reduction of X is X itself.

dependent, i.e. it gives different results depending on the order in which the variables appear.

Unfortunately, there are some problems with the concept of reduction. Very few probability bases are reducible by 100%, only precisely those bases where the arithmetical midpoint of the hull in each dimension is a consistent point. The random chance for this to occur is very slim. For other bases, reduction might be possible to any extent in the interval [0%, 100%]. Although a base reducible by 0% is clearly undesirable, it is not clear that a base with an 80% possible reduction should be inferior in any respect to a 100% reducible base. In effect, the reduction procedure often dismisses all consistent points in a base, keeping only inconsistent points. Even worse is that the reduction is not invariant when the same decision model is refined. Two examples show the counter-intuitive behaviour of the reduction operator.

Example 2: Consider the following probability base P_1:

$p_{11} \in [0.00, 1.00]$

$p_{12} \in [0.00, 0.40]$

By the normalisation $\sum_i p_{1i} = 1$, the orthogonal hull is

$p_{11} \in [0.60, 1.00]$

$p_{12} \in [0.00, 0.40]$.

This base is reducible by 100% to

$p_{11} \in [0.80, 0.80]$

$p_{12} \in [0.20, 0.20]$

which is a reasonable result. The same situation can be modelled in more detail by splitting the consequence C_{12} into two consequences C_{22} and C_{23}.

$p_{21} \in [0.00, 1.00]$

$p_{22} + p_{23} \in [0.00, 0.40]$

By the normalisation $\sum_i p_{2i} = 1$, the orthogonal hull is

$p_{21} \in [0.60, 1.00]$

$p_{22} \in [0.00, 0.40]$

$p_{23} \in [0.00, 0.40]$.

This base is only reducible by 67% to

$p_{21} \in [0.73, 0.87]$

$p_{22} \in [0.13, 0.27]$

$p_{23} \in [0.13, 0.27]$

after which there is no consistent point left in the base. This argument can continue with finer splits of C_{12} and for each split, the possible reduction decreases. A base with a split to four consequences is only reducible by 40%, and to eight consequences only by 20%. Since the same situation is being modelled, it is not satisfying to see the reduction, and thus the proportion, go down closer to zero for modelling situations that should have the same result. Thus, the reduction operation is not invariant with respect to the model.

Example 3: Next consider another probability base P_3. Suppose it contains the following statements:

$p_{1i} \in [0.00, 0.25]$ for $i \in \{1,\ldots,5\}$.

By the normalisation $\sum_i p_{1i} = 1$, the orthogonal hull is the same as the statements themselves. This base is reducible by 40% to

$p_{1i} \in [0.05, 0.20]$ for $i \in \{1,\ldots,5\}$.

If the decision maker decides to make the intervals a little wider, say by 5%, he will have the base P_4 with

$p_{1i} \in [0.00, 0.30]$ for $i \in \{1,\ldots,5\}$.

The orthogonal hull is again equal to the statements themselves. This base is reducible by 67% to

$p_{1i} \in [0.10, 0.20]$ for $i \in \{1,\ldots,5\}$.

By the same reasoning, making the intervals 5% wider again will result in the base being reducible by 86%. It is not intuitively clear for a decision maker what this would mean and what information he might receive from those figures. For a decision situation with ten consequences, the results are even more remarkable. It is not clear what correspondence the possible reductions have to the real merits of an alternative.

For utility bases, the situation is different but no less troublesome. The main differences are the absence of compound statements (in their formulation) and the absence of a normalisation constraint (by definition) for each alternative. Hence, utility bases without comparisons are always reducible by 100% while bases with comparisons are often not. This should not be considered to imply that utility bases without comparisons are "better" in any respect. On the contrary, a base with many comparative statements often contains more useful ordering information that yields clearer results when comparing the alternatives.

The Concept of Proportion

Based on the reduction procedure the concept of proportion measures how far a sequence of G-reductions can be carried out in a base while maintaining consistency.

> **Definition 2:** X is a consistent base. The *proportion* of the base X is the number PROP(X) $\in [0, 1]$ such that PROP(X) $=_{def}$ sup(ρ: $\rho \in [0, 1]$ and the G(ρ)-reduction of X is consistent).

To see how the proportion can be used to discriminate between more than one alternative being t-admissible, consider the proportion of the combined base $P \cup U$ in a decision frame when the statement $\delta_{ij} \geq -t$ is added.

> **Definition 3:** Given a decision frame F with a probability base P and a utility base U. ${}^{t}\text{PROP}_{ij}(PU) =_{def} \text{PROP}(\{\delta_{ij} \geq -t\} \cup P \cup U)$ is the proportion of the alternative A_i compared to A_j at level t.

Due to the definition of δ_{ij} the values ${}^{t}\text{PROP}_{ij}(PU)$ and ${}^{t}\text{PROP}_{ji}(PU)$ need not be the same, nor sum to one or to any other constant.

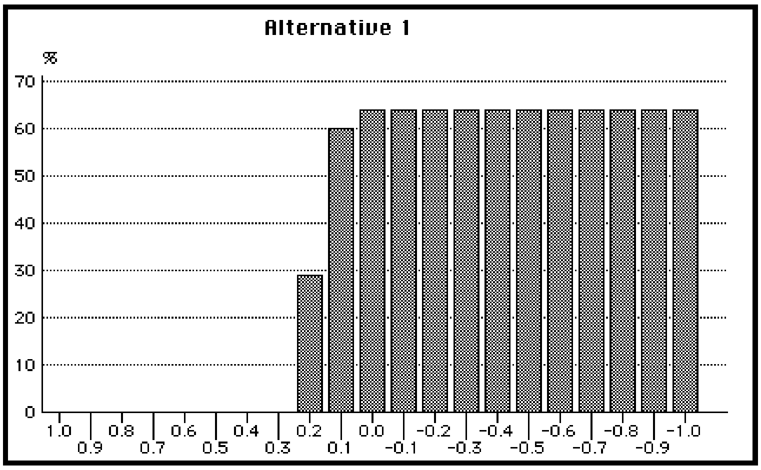
Example 4: Consider a decision situation with two alternatives. The alternative A_1 has the consequences C_{11}, C_{12}, and C_{13}, and A_2 has the consequences C_{21} and C_{22}. The utility base contains the following statements.

$u_{11} = 1.00$
$u_{12} = 0.00$
$u_{13} = 0.00$
$u_{21} = 0.40$
$u_{22} = 0.40$

In the probability base, the consequence C_{11} has the probability $p_{11} \in [0.00, 0.67]$. For the probabilities of all the other consequences, only the normalisation constraints apply. An evaluation of the two alternatives will result in the following proportions. For all t-values down to t = 0 both alternatives have the proportions 67%. For t = –0.1 both have 60%. The next two proportions differ and they are the values that discriminate the alternatives. For t = –0.2 A_1 has a proportion of 30% while A_2 has 40%. Likewise, for t = –0.3 A_1 is worse with 0% where A_2 reaches 20%. From t = –0.4 downwards, none of the alternatives have any proportion at all. This can be displayed as in Figure 1.[4] The bars of a recommendable

[4] The x-axis shows t-values but with opposite signs. Thus, larger horizontal axis values indicate a stronger alternative.

alternative should be tall as far left as possible. Here, A_2 is clearly favourable to A_1.



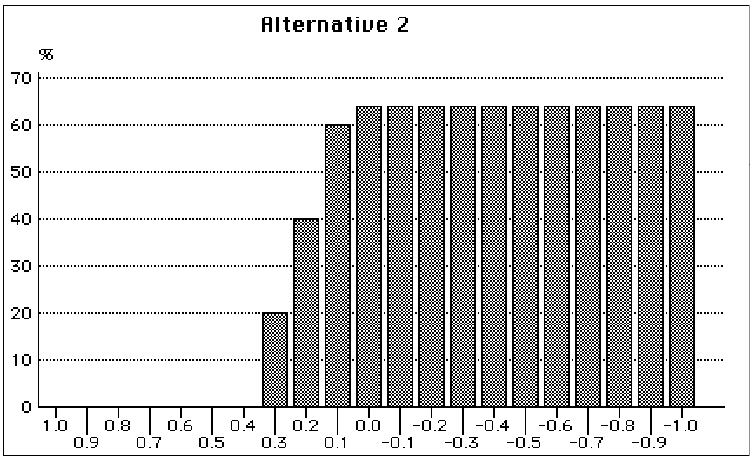


Figure 1. Comparing the proportions of two alternatives (from the μ software)

Definition 4: Given a decision frame F with m alternatives. F contains a probability base P and a utility base U. An alternative A_i is *preferable* to A_j **iff** for all $t \in [-1, 1]$ ${}^{t}\mathrm{PROP}_{ij}(PU) > {}^{t}\mathrm{PROP}_{ji}(PU)$.

Definition 5: Given a decision frame F with m alternatives. F contains a probability base P and a utility base U. Two alternatives A_i and A_j are *equally preferable* **iff** for all $t \in [-1, 1]$ ${}^{t}\mathrm{PROP}_{ij}(PU) = {}^{t}\mathrm{PROP}_{ji}(PU)$.

These two concepts are used to discover preferable alternatives. But what if one alternative has taller bars for higher t-values[5] and the other one has

[5] Weaker tests.

taller bars when both are falling for lower t-values[6] – which alternative should then be recommended? This question points to the simplest form of the *t/proportion duality*; one alternative has larger proportions but falls faster for stronger tests, and the other one has smaller proportions but falls slower for stronger tests. In [E94] it is shown under which circumstances this occurs, and it is from there that Example 5 is taken. The conclusion is that no general criteria exist for comparing such alternatives, and only rules of thumb are left for the simplest situations. Comparing two alternatives A_i and A_j, one of three situations may occur.

(i) One of A_i and A_j is preferable
(ii) A_i and A_j are equally preferable
(iii) A_i and A_j are t/proportion dual

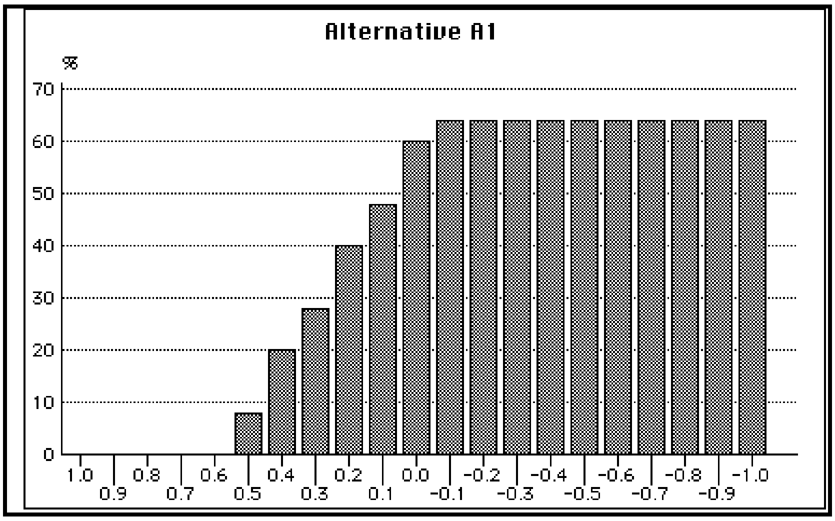
For case (iii), when the two definitions of preferable above fail for some t-value, nothing can be said about which alternative to choose. It is not clear that considering criteria such as the total bar area displayed for a certain alternative would help the decision-maker.

Example 5: Consider a decision situation, almost similar to the previous example, with two alternatives. A_1 has the consequences C_{11}, C_{12}, and C_{13} and A_2 has the consequences C_{21} and C_{22}. The utility base contains the same statements as above:

$u_{11} = 1.00$
$u_{12} = 0.00$
$u_{13} = 0.00$
$u_{21} = 0.40$
$u_{22} = 0.40$

This time, there are no decision-maker statements of probability. Only the normalisation constraints apply to the probability variables. If the alternatives are compared according to the proportion method, the results will be as in Figure 2. The proportions for alternative A_1 do not fall as fast for lower t-values. Since the utility statements are pointwise statements in the example for the sake of simplicity, the evaluation can easily be discussed analytically.

[6] Stronger tests.



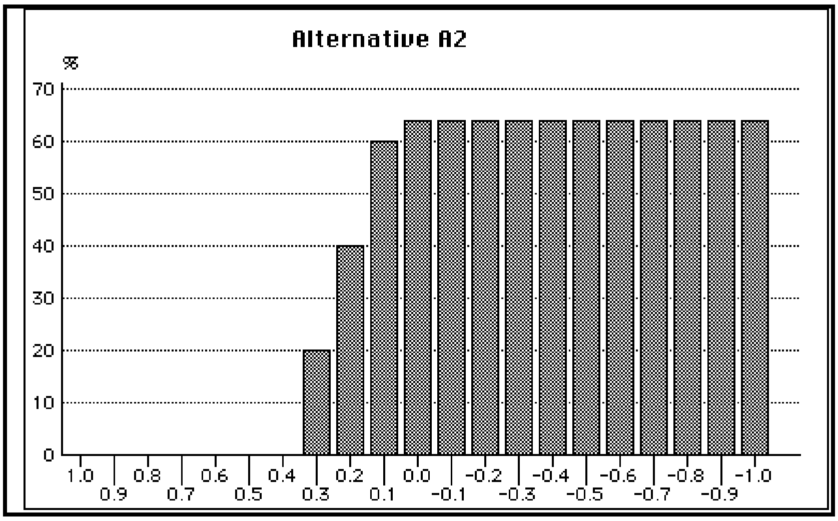


Figure 2. t/proportion duality when comparing two alternatives (also from μ)

For a G(α)-reduction, the probability variables become

$p_{1i} \in [\alpha/2, 1-\alpha/2]$ for i = {1,2,3}

$p_{2i} \in [\alpha/2, 1-\alpha/2]$ for i = {1,2}

as long as $1.5 \cdot \alpha \leq 1$, after which the base becomes inconsistent and thus impossible to work with.

It is easy to obtain the ranges for the expected utilities.

$E_1 \in [\alpha/2, 1-\alpha]$

$E_2 \in [0.40, 0.40]$

This leads to determining $\delta_{ij} > -t$ for the range $t \in [-1, 1]$. Consider a few sample values of t and try to make a maximal G(α)-reduction. Here, PU' refers to $P \cup U \cup \{\delta_{ij} > -t\}$ for appropriate i and j.

$t = -0.4$: $\delta_{12} > -t$, ${}^{PU'}\max(\alpha) = 0.20 \Rightarrow {}^{-0.4}\text{PROP}_{12} = 20\%$
$\delta_{21} > -t$, ${}^{PU'}\max(\alpha) = 0.00 \Rightarrow {}^{-0.4}\text{PROP}_{21} = 0\%$
$t = -0.1$: $\delta_{12} > -t$, ${}^{PU'}\max(\alpha) = 0.50 \Rightarrow {}^{0.1}\text{PROP}_{12} = 50\%$
$\delta_{21} > -t$, ${}^{PU'}\max(\alpha) = 0.60 \Rightarrow {}^{0.1}\text{PROP}_{21} = 60\%$

The example clearly demonstrates that it is a t/proportion situation. Already these two points exhibit the duality property. Even in this oversimplified case, it is not clear which alternative is the best. In more realistic problems, duality can become a complicated issue.

Figure 2 above shows an information base with several alternatives of which one has three consequences and where the only probability statement for that alternative is $\sum_j p_{ij} = 1$. Such a base is bound to become inconsistent for proportions above 67%. Thus, for those t-values, the method delivers no information.

The concept of proportion inherits the deficiencies of the reduction operator. The proportion expresses how much a base can be reduced while still maintaining $\delta_{ij} \geq -t$, but a failure can have more than one cause. Assume that there is a consistent probability base P that is to be $G(\alpha)$-reduced. Then *either*

(i) the $G(\alpha)$-reduction itself fails for the base P, *or*
(ii) the $G(\alpha)$-reduction is possible for the base P, but $\delta_{ij} \geq -t$ is inconsistent with P.

In case (ii) at least the algorithm has found some kind of quality measure, but in case (i) the base is not centred enough to be reduced in this way. Even worse, if more than two alternatives are compared, the statements concerning a third alternative might make the base P impossible to reduce beyond, say, 15%. Then the proportion of any alternative cannot reach any higher than 15%.[7] As more alternatives with more consequences are added, the chance of being exposed to this deficiency increases. This clearly demonstrates that the proportion cannot be employed to determine which alternative constitutes the best choice.

[7] If none of the compound or comparative statements involve probability variables from different alternatives, the dependency on a third alternative can be remedied by reducing the probability base separately for each alternative. However, this does not save the concept of proportion.

References

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