When a rigid-body rotates (with possibly time-variable orientation and angular velocity) its orientation, angular velocity, angular momentum and moment of inertia tensor are coordinate dependent, and may be given in two distinct frames of reference:

- inertial, space or world frame of reference
- non-inertial, body-fixed or rotational frame of reference

Equation (7.47)

Here, the time derivative of the orientation quaternion is expressed in *inertial* (world) frame of reference as

$$\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t} = \mathbf{T}(q)\,\boldsymbol{\omega} \quad \text{Eq. (7.47)} \quad \Leftrightarrow \quad \boxed{\dot{q} = \frac{1}{2}\omega^{\text{world}} q}$$

where $\omega = (0, \boldsymbol{\omega})$ is angular velocity quaternion in world coordinates (inertial frame) and $\mathbf{T}(q)$ is a matrix according to (7.48) that corresponds to ½ of the right side multiplication with quaternion q, i.e. since

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{pmatrix} \cdot \begin{pmatrix} \omega \mathrm{w} \\ \omega \mathrm{x} \\ \omega \mathrm{y} \\ \omega \mathrm{z} \end{pmatrix} = \begin{pmatrix} w\,\omega\mathrm{w} - x\,\omega\mathrm{x} - y\,\omega\mathrm{y} - z\,\omega\mathrm{z} \\ x\,\omega\mathrm{w} + w\,\omega\mathrm{x} + z\,\omega\mathrm{y} - y\,\omega\mathrm{z} \\ y\,\omega\mathrm{w} - z\,\omega\mathrm{x} + w\,\omega\mathrm{y} + x\,\omega\mathrm{z} \\ z\,\omega\mathrm{w} + y\,\omega\mathrm{x} - x\,\omega\mathrm{y} + w\,\omega\mathrm{z} \end{pmatrix} = \omega q$$

which holds for any two quaternions $\omega = (\,\omega\mathrm{w}, \omega\mathrm{x}, \omega\mathrm{y}, \omega z\,)$ and $q = (\,w, x, y, z\,)$.

Similarly, the time derivative of the orientation can be expressed in a *body-fixed* frame as

$$\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{2}\omega^{\text{world}} q = \frac{1}{2}(q\,\omega^{\text{body}} q^{*})q = \frac{1}{2} q\,\omega^{\text{body}} \quad \text{i.e.} \quad \boxed{\dot{q} = \frac{1}{2} q\,\omega^{\text{body}}}$$

Equation (7.56), Euler's equation

This equation is the major source of confusion, since Euler's equation is actually specified in **body-fixed** (non-inertial frame, see [1]), as it follows from the general proposition

$$\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)^{\text{world}} = \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)^{\text{body}} + \boldsymbol{\omega}^{\text{body}} \times \mathbf{A}^{\text{body}}$$

which is valid for any vector $\mathbf{A}$ (also for $\mathbf{L}$ in particular). Thus, Eq. (7.56)

$$\dot{\mathbf{L}} = \mathbf{I}\,\boldsymbol{\omega} + \boldsymbol{\omega} \times (\mathbf{I}\,\boldsymbol{\omega}) = \boldsymbol{\tau}$$

means

$$\dot{\mathbf{L}}^{\text{world}} = \dot{\mathbf{L}}^{\text{body}} + \boldsymbol{\omega}^{\text{body}} \times \mathbf{L}^{\text{body}} = \boldsymbol{\tau}^{\text{world}}$$

i.e.

$$\boxed{\dot{\mathbf{L}}^{\text{body}} = \boldsymbol{\tau}^{\text{world}} - \boldsymbol{\omega}^{\text{body}} \times (\mathbf{I}^{\text{body}} \boldsymbol{\omega}^{\text{body}})}$$

where $\mathbf{I}^{\text{body}} = \mathbf{I}'$ is principal moment of inertia, which is ***constant*** in the body-fixed frame.

To make all this more clear, different flavors of the algorithm for rigid body motion in 3D are summarized in the following table, depending whether $\boldsymbol{\omega}$, $\mathbf{L}$ or $\mathbf{I}$ are specified in the world or the body-fixed reference frame, and whether gyroscopic effect is accounted for (included) or ignored (ignored as to increase stability of the algorithm for low-order integrator methods).

Inertial (space or world) frame		Body-fixed (rotational) frame	
with gyroscopic effect included	with gyroscopic effect ignored	with gyroscopic effect included	with gyroscopic effect ignored
Linear part (specified always in inertial frame since attached to barycenter that does not rotate):			
$\mathbf{p}_{n+1} = \mathbf{p}_n + h\,\mathbf{f}_n$ and $\mathbf{x}_{n+1} = \mathbf{x}_n + h\,\mathbf{v}_{n+1}$ where the velocity is derived quantity $\mathbf{v}_n = m^{-1}\mathbf{p}_n$			
Angular part (frame dependent):			
$\mathbf{L}_{n+1} = \mathbf{L}_n + h\,\boldsymbol{\tau}_n$	$\mathbf{L}_{n+1} = \mathbf{L}_n + h\,(\boldsymbol{\tau}_n + \boldsymbol{\omega}_n \times \mathbf{L}_n)$	$\mathbf{L}_{n+1} = \mathbf{L}_n + h\,(\boldsymbol{\tau}'_n - \boldsymbol{\omega}_n \times \mathbf{L}_n)$	$\mathbf{L}_{n+1} = \mathbf{L}_n + h\,\boldsymbol{\tau}'_n$
$\dot{q}_{n+1} = \frac{1}{2}\omega_{n+1}\,q_n$ where $\omega_{n+1} = (0, \boldsymbol{\omega}_{n+1})$		$\dot{q}_{n+1} = \frac{1}{2}q_n\,\omega_{n+1}$ where $\omega_{n+1} = (0, \boldsymbol{\omega}_{n+1})$	
$q_{n+1} = q_n + h\,\dot{q}_{n+1}$			
$\boldsymbol{\tau}_n$ is in inertial frame		$\boldsymbol{\tau}'_n$ in body-fixed frame is $\boldsymbol{\tau}'_n = q^*\,\boldsymbol{\tau}_n\,q$	
Derived quantities:			
$\boldsymbol{\omega}_n = \mathbf{I}_n^{-1}\mathbf{L}_n$		$\boldsymbol{\omega}_n = \mathbf{I}_n^{-1}\mathbf{L}_n$	
$\mathbf{I}_n^{-1} = \mathbf{R}(q_n)\,\mathbf{I}'^{-1}_n\,\mathbf{R}(q_n)^{\mathrm{T}}$		$\mathbf{I}_n^{-1} = \mathbf{I}'^{-1}_n = const$	
$E_n^{\mathrm{k}} = \frac{1}{2}m^{-1}\mathbf{p}_n^2 + \frac{1}{2}\mathbf{I}_n^{-1}\mathbf{L}_n^2$			
Notes: To transform any vector $\mathbf{x}'$ from body-fixed frame to $\mathbf{x}$ in inertial frame do $\mathbf{x} = q\,\mathbf{x}'\,q^*$. E.g. the relation between the total torque $\boldsymbol{\tau}_n$ in inertial frame and $\boldsymbol{\tau}'_n$ in body-frame is $\boldsymbol{\tau}_n = q\,\boldsymbol{\tau}'_n\,q^*$. Note also that the orientation quaternion q is the same in both frames of reference, since $q = q\,q\,q^*$.			

References:

[1] Taylor, John R. – *Classical Mechanics*, University Science Books, 2005, pp. 394-396

[2] Severin, M. – *Notes on Discrete Mechanics*, April 18, 2011